Probability Teasers

1. A family has 2 children. We pick one child and find that he is a boy. Thus, at least one of them is a boy. **What is the probability that the other child is a boy, given one is a boy? ____**

For each of the possible answers below, indicate whether the logic is true or false with a "T" or "F." Put a T/F next to each one as you read it and write a brief justification. In the answers, the symbols B=boy, G= girl, and B1 = one child is a boy.

a) There are four possible combinations of children (BB, GG, BG, and GB), and only one has 2 boys. The answer is .25.

b) A family can have either all boys, all girls, or one of each. With a boy, there are three possibilities left, so having both boys has a .33 chance of happening.

c) Since there are 4 possible combinations, and one of them (GG) can't exist given one child is a boy, that leaves three possibilities. Only one of the three is all boys, so the answer is .33.

d) Since there are 3 types of families, all-boy, all-girl, and mixed, and the all-girl one can't exist given that one child is a boy, there are 2 possibilities only one of which has both boys. The answer is therefore 1 out of 2 or .50.

e) Since births of children are independent of each other, the first child's sex has no bearing on the other's. Since about half of the births are male, the probability of the second child being a boy is .50.

f) If B1 = first child is a boy, B2 that the second is a boy, then, by the independence rule and the formula for conditional probabilities, $p(B2 \mid B1) = p(B2) = .50$.

g) Because births are independent, $p(B1 \& B2) = p(B1)p(B2) = (.5)(.5)=.25$.

h) The probability of the other child being a boy given one is is conceptually identical to the event of both children being boys given the first one is. We want to know the $p(BB \mid B1)$, the probability of 2 boys given one is.
By Bayes' formula,

$$p(BB \mid B1) = \frac{p(B1 \mid BB)\, p(BB)}{p(B1)} = \frac{(1.0 \times .25)}{.50} = .50$$

i) Answer h) is wrong because the probability of having <u>at least one boy</u> is not .50. With the four possible outcomes of two-children families, three have at least one boy (BB, BG, GB) and one does not (GG). Thus, p(B1) is .75. Replacing .50 with .75 in Bayes' formula gives .333; one-third is the answer.

j) Since at least one of the children is a boy, the probability of the other one being a boy is .33. There are 3 ways at least one of them could be a boy: BB, BG, GB. But only one of them has 2 boys.

The ANSWERS

Answer a) is wrong because it determines the probability of 2 boys, not the probability that both are boys given one is. Thus, this answer leaves out some important information-- the fact that you know one child is a boy. This information destroys the "equally likely" assumption. You can easily see that the 4 possible outcomes are **no longer** equally likely when you realize that the information that one child is a boy makes the probability of the event GG = zero.

Answer b) is wrong because of the failure to consider the "equally likely" assumption. The three events (all boys, all girls, one of each) are not equally likely, so we cannot divide the total probability equally among them. There are two ways you can have one of each sex, GB and BG. The other two possible outcomes are BB and GG. Thus, the probability of one of each sex .50, all boys is .25, and all girls .25. However, these probabilities do not answer the question. The question does not ask for the probability of 2 boys, it asks the probability of 2 boys **given that one is**. .

Answer c) is almost correct. What is lacking is the realization that the given-one-boy information not only changes the p(GG) from .25 to .00 but also changes the probability of the remaining three outcomes so that they are no longer equally likely. In the GB and the BG families, there is only one way for the one child we observed to be a boy, but there are two ways in a BB family for us to see one boy (there are two different boys we could have selected first). Thus, observing one boy makes it twice as likely that we have a BB family than it does a mixed sex family.

Another way to reason correctly with this method is redefine GB as "pick a girl first, then a boy." We then define BG as "pick a boy then a girl" and BB as pick a boy and then another one. Now you can realize that observing the first child makes the GB family impossible as well as the GG family. With this method we see that the information now leaves us with 2 equally possible outcomes, BG and BB, so the answer is .50.

Answer d) arrives at the correct answer for the wrong reason. It does take into account the information about the one boy child, but it mistakenly assumes that the remaining two options (an all-boy and a one-boy family) are always equally likely. To see that that they are not, consider a three-child family. Such a family can have all boys, two boys, one boy or no boys. These 4 events are not equally likely so the probability of each one is **not** 1/4 or .25 . There is only one way to have an all-boy family, but there are three ways to have a one-boy family (BGG, GBG, GGB). The observation of one boy child does eliminate the all-girl possibility, but the remaining three events (all boys, one boy, two boys) are not equally likely. The probability of having an all-boy, three-child family would **not** be one-third; it is actually one-fourth (having observed the first child to be a boy leaves leaves BGG, BGB, BBG, and BBB as equally likely possibilities).

Answer e) uses logic or reasoning to arrive at the right answer. It is always good to try this method. However, some questions involve too many possible outcomes for us to consider in our heads. The formulas used in answers f) and h) are formal ways to express logical relationships; learning to employ them will give us answers to problems too complicated to reason through in our heads. Also, sometimes our natural language means of thinking is not precise enough for mathematics. Some semantic terms do not map unambiguously into probability values.

Answer f) uses logic and the formulas to arrive at the correct answer in an easy way.

Answer g) is wrong for the same reasons as was answer a). However, this answer at least tries to use a formal, abstract formula.

Answer h) is correct, although the means of arriving at the answer is more complicated than in answer f). The term, p(B1 | BB), equals 1 because, given the family is an all boy one, any child picked will always be a male. The term "p(BB)" = .25 because only 1 out of 4 two-child families will have all boys by the reasoning in answers a) or g).

Answer i) is wrong about what p(B1) represents. The probability of one child being male does **not** equal the probability of at least one child being male. Answer i) misinterprets the question to be "probability of other child being male given *at least one* is a male" when the question asks "given one child is male." Answer i) is correct about the probability of *at least one* child being male being .75, but that is not what p(B1) represents. Although the distinction between one child being male and at least one child being male seems trivial or nitpicking, it is not mathematically. Also note that the event "one child being male" is not the same thing as "exactly one child being male" or "only one child being male." Our natural language or semantics frequently are not precise enough to express important distinctions in probability. This is another reason why you should learn to use the formulas as opposed to only verbally thinking through the problems.

Answer j) has the correct logic, but it has misinterpreted the question. The conditioning event (the information given) is "one is a boy." This answer misinterprets the conditioning even to be "at least one is boy." This subtle distinction is explained in analysis of answer i above.

Answers a-c specify the entire sample spaces, element by element, and then form ratios of the number of events divided by the total number of elements to obtain a probability. This is very laborious when there are many elements. Think how long it would take to answer the following question: A family has 10 children. The first three children are male. What is the probability that 6 of the remaining 7 children are male? The conditional and Bayesian formulae, used in e) and g), are short cuts for listing the entire sample space. Learning to use them will save you a lot of time.